6.4.2. Reconstruction of Signal form Samples

Let us consider equation (*ix*) as

$$x(t) = IFT\left\{\frac{1}{2f_m}\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right)e^{-j\pi fn/f_m}\right\}$$

or
$$x(t) = \int_{-f_m}^{f_m}\frac{1}{2f_m}\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right)e^{-j\pi fn/f_m}e^{j2\pi ft}\,dt$$

Interchanging the order of summation and integration, we get

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right)\frac{1}{2f_m}\int_{-f_m}^{f_m} e^{j2\pi f\left(t-\frac{n}{2f_m}\right)}\,dt$$

or
$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right)\frac{\sin(2\pi f_m t - n\pi)}{(2\pi f_m t - n\pi)}$$

or
$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right)\frac{\sin\pi(2f_m t - n\pi)}{\pi(2f_m t - n)}$$

Since,
$$\text{sinc}\,\theta = \frac{\sin\pi\theta}{\pi\theta}$$

Therefore,
$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right)\text{sinc}\,(2f_m t - n)\quad -\infty < n < \infty$$

Hence, this is the interpolation formula to reconstruct $x(t)$ from its samles $x(nT_s)$. Therefore, from all above, it is clear that the signal may be completely represented into and recovered from its samples if the spacing between the successive samples is $\frac{1}{2f_m}$ seconds *i.e.*, $f_s = 2f_m$ samples per second.

Sampling Frequency for Bandpass Signal :

Since the spectral range of the bandpass signal is 20 kHz to 82kHz Therefore

$$\text{Bandwidth} = 2f_m$$
$$= 82\text{ kHz} - 20\text{ kHz} = 62\text{ kHz}$$

Hence, Minimum Sampling rate = 2 × bandwidth

$$= 2 \times 62 = 124 \text{ kHz}$$

Generally, the range of minimum sampling frequencies is specified for band-pass signals.

It lies between $4f_m$ to $8f_m$ samples per second.

Therefore,

Range of minimum sampling frequencies

$$= (2 \times \text{bandwidth}) \text{ to } (4 \times \text{bandwidth})$$

$$= 2 \times 62 \text{ kHz to } 4 \times 62 \text{ kHz} = 124 \text{ kHz to } 248 \text{ KHz}$$ **Ans.**

6.5. Impulse Train Sampling of a Continuous-time Signals

As discussed in last article the sampling theorem may be developed by using a convenient method in which we represent the sampling of a continuous-time signal at regular interval. A useful method to perform sampling is through the use of periodic impulse train multiplied by continuous-time signal $x(t)$ which is desired to be sampled. This type of sampling is known as **impulse-train sampling.** Figure 6.6 explains the mechanism of impulse train sampling of a continuous-time signal $x(t)$.

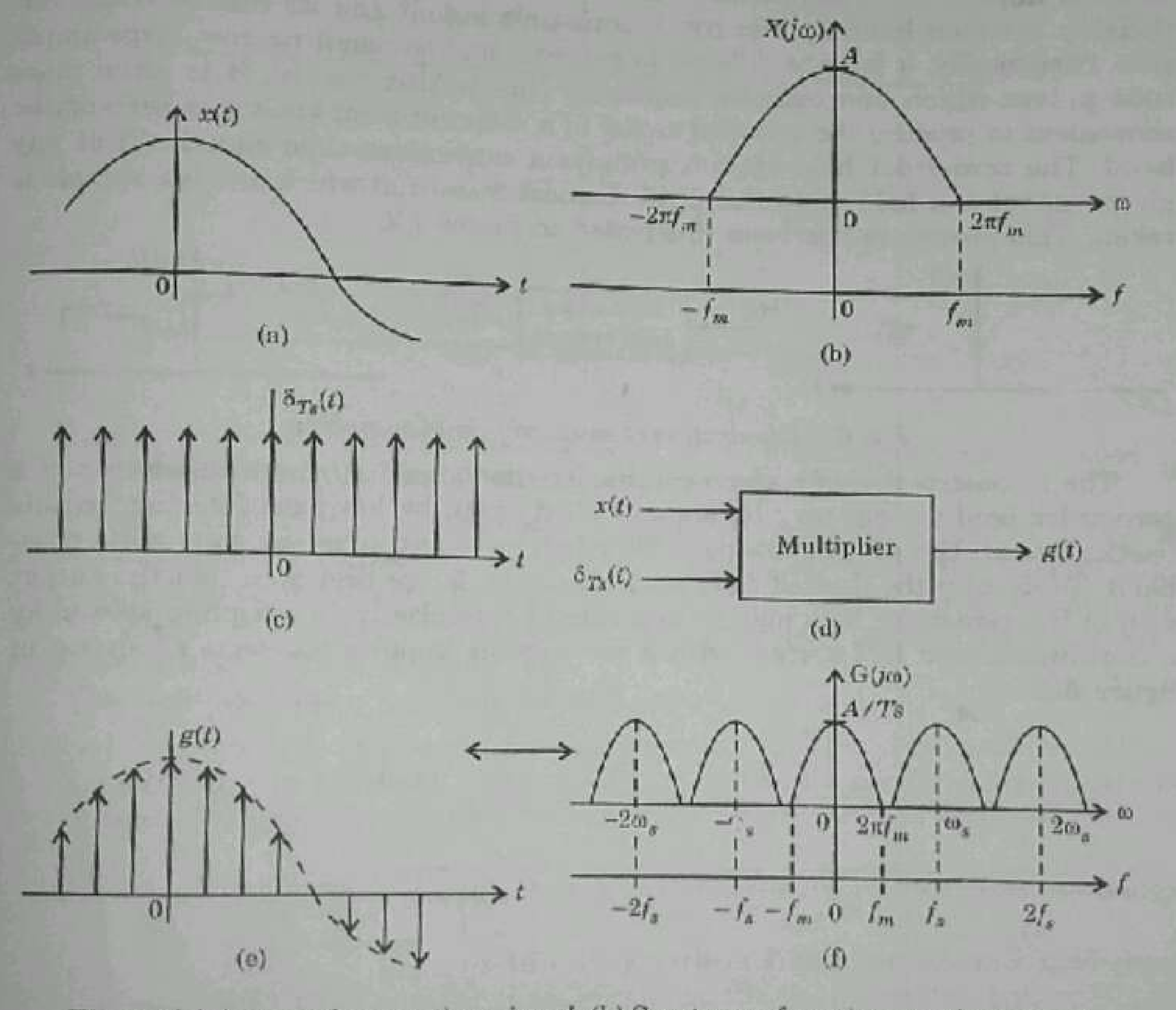

Fig. 6.6(a) Any continuous-time signal, (b) Spectrum of continuous-time signal, (c) Impulse train as sampling function, (d) Multiplier, (e) Sampled signal, (f) Spectrum of sampled signal.

The periodic impulse train $\delta_{T_s}(t)$ is called the sampling function. The sampling period of a periodic impulse train is denoted by T_s. Also, the reciprocal of sampling period T_s is known as sampling frequency f_s. Thus, the fundamental frequency of the periodic impulse train $\delta_{T_s}(t)$ is given as

$$\omega_s = \frac{2\pi}{T_s} = 2\pi f_s.$$

This is also known as angular sampling frequency.

Thus as discussed in last article sampled signal in time-domain is given as

Also, the sampled signal in frequency domain is expressed as

$$g(t) = x(t)\,\delta_{T_s}(t)$$

where $x(t)$ = continuous-time signal

and $\delta_{T_s}(t)$ = periodic impulse train.

$$G(j\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(j\omega - n\,j\omega_s) \qquad ...(6.10)$$

6.6. Zero-Order Hold Sampling

As discussed in article (6.3), the sampling theorem may be easily described in terms of impulse-train sampling. In fact, the sampling theorem establishes a relationship between band-limited continuous-time signal and its discrete-time version. Paractically, it is quite difficult to generate and transmit narrow, large-amplitude pulses which approximate impulses. Due to this reason, it is often more convenient to produce the sampled signal in a different form known as **zero-order hold**. The zero-order hold system samples a continuous-time signal $x(t)$ at any given instant and hold this value until the next instant at which another sample is taken. This procedure has been illustrated in figure 7.7.

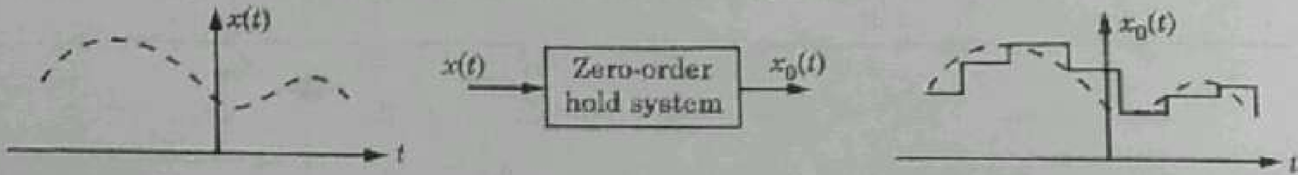

Fig. 6.7. Illustration of zero-order hold sampling.

The reconstruction of a given continuous-time signal $x(t)$ from the output of a zero-order hold system may be accomplished again by low-pass filtering. In this particular cas, the desired low-pass filter has no longer constant gain in its pass-band. To develop the desired filter characteristics, let us first note that the output $x_0(t)$ of the zero-order hold may be generated by inpulse train sampling follwed by a continuous-time LTI system with a reetangular impulse response as shown in figure 6.8.

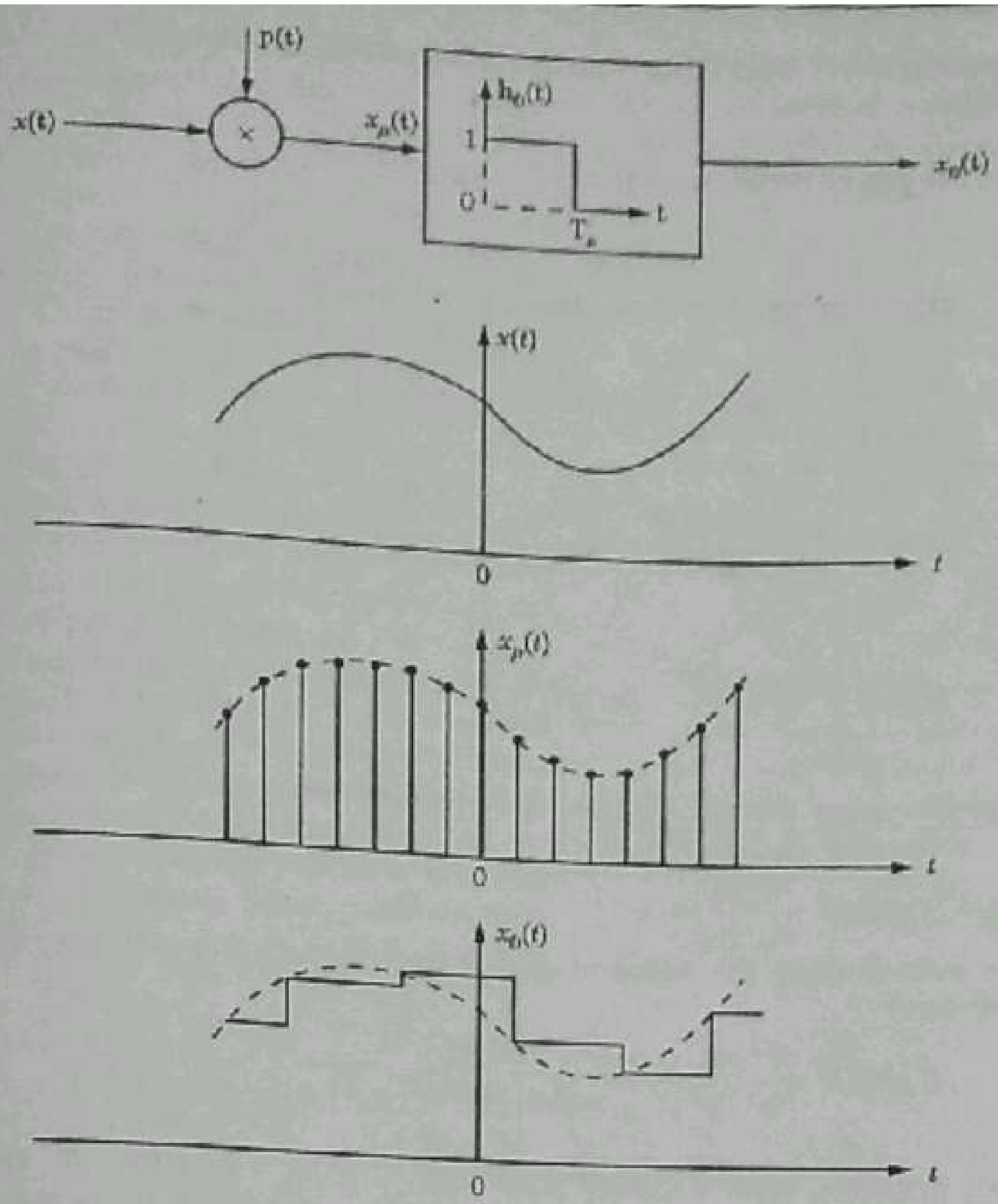

Fig. 6.8. Illustration of a zero-order hold system as impulse-train sampling followed by an continuous-time LTI system with a rectangular impulse response.

Thus, to construct a continuous-time signal $x(t)$ from the output of the zero-order hold $x_0(t)$, we consider processing $x_0(t)$ with continuous-time LTI system with impulse-response $h_r(t)$. Its frequency-response then would be denoted by $H_r(j\omega)$. Figure 6.9 shows the cascade of zero-order hold system with a reconstruction low-pass filter of figure 6.8.

The impulse response $h_0(t)$ is defined as

$$h_0(t) = \begin{cases} 1, & 0 \le t \le T_s \\ 0, & \text{otherwise} \end{cases} \qquad \text{...(6.11)}$$

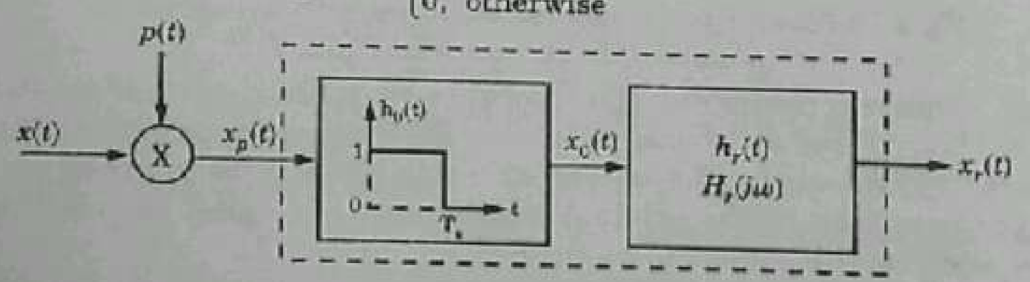

Fig. 6.9. Cascade combinatopn of zero-order hold system with a reconstruction filter.

Frequency response $H_0(j\omega)$ may be determined by taking CTFT of the impulse response $h_0(t)$ as

$$H_0(j\omega) = \text{CTFT}\,[h_0(t)] = \int_{-\infty}^{\infty} h_0(t)\, e^{-j\omega t} dt \quad ...(6.12)$$

or $$H_0(j\omega) = \int_0^{T_s} 1.\, e^{-j\omega t} dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^{T_s} = \frac{e^{-j\omega T_s} - e^{-j\omega(0)}}{-j\omega} = \frac{e^{-j\omega T_s} - 1}{-j\omega}$$

or $$H_0(j\omega) = \frac{e^{-j\omega(T_s/2)}}{\omega}\left[\frac{e^{-j\omega Ts/2} - e^{-j\omega Ts/2}}{-j}\right]$$

$$= \frac{2e^{-j\omega(Ts/2)}}{\omega}\left[\frac{e^{j\omega Ts/2} - e^{-j\omega Ts/2}}{2j}\right]$$

or $$H_0(j\omega) = \frac{2e^{-j\omega Ts/2}}{\omega}.\sin\left(\frac{\omega Ts}{2}\right) = e^{-j\omega Ts/2}\left[\frac{2\sin\left(\frac{\omega Ts}{2}\right)}{\omega}\right] \quad ...(6.13)$$

Thus, frequency response $H_r(j\omega)$ may be determined as

$$H(j\omega) = H_0(j\omega).\, H_r(j\omega)$$

or $$H_r(j\omega) = \frac{H(j\omega)}{H_0(j\omega)} \quad ...(6.14)$$

Now, substituting the value of $H_0(j\omega)$ from equation (6.13), into equation (6.14), we have

or $$H_r(j\omega) = \frac{H(j\omega)}{H_0(j\omega)} = \frac{H(j\omega)}{e^{-j\omega Ts/2}\left[\frac{2\sin \omega Ts/2}{\omega}\right]} = \frac{e^{j\omega Ts/2}.\, H(j\omega)}{\left(\frac{2\sin \omega Ts\ /2}{\omega}\right)} \quad ...(6.15)$$

As an example, with a cut-off frequency of $H(j\omega)$ equal to $\omega_s/2$, the ideal magnitude and phase for the reconstruction low-pass filter following a zero-order hold has been shown in figure 6.10.

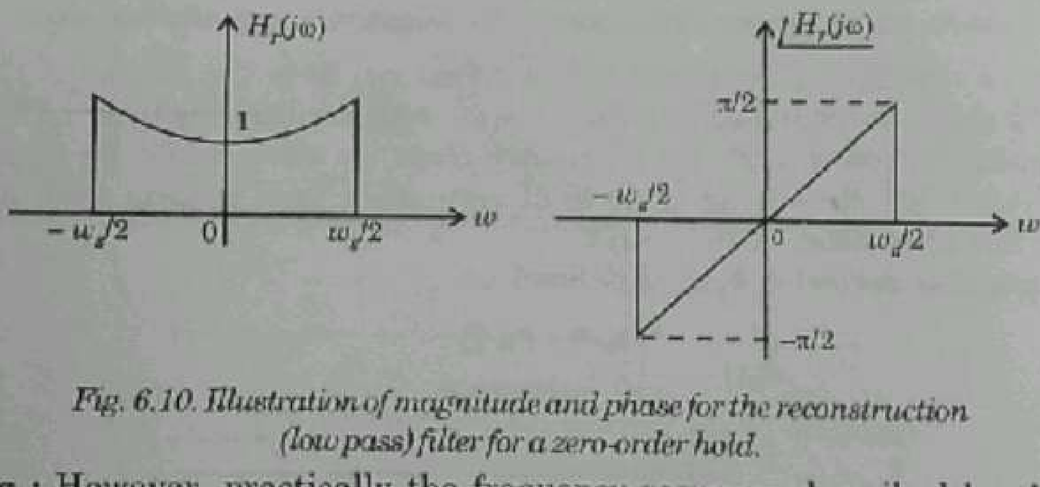

Fig. 6.10. Illustration of magnitude and phase for the reconstruction (low pass) filter for a zero-order hold.

Note : However, practically the frequency response described by the equation can not be realized and hence we design an adequate approximation to it. In fact, in several situations, the output of a zero-order hold is taken as an adequate approximation to the original continuous-time signal by itself, without using any additional low-pass filter (LPF).

Interpolation is one such method which is used to recover a continuous-time signal from its samples.